The Transmission of Electric Waves round the Earth.

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On the hypothesis that the Earth consists of an imperfectly conducting sphere surrounded by infinite homogeneous dielectric, I have recently* obtained a complete solution (in a form adapted for numerical computation) of the problem of determining the effect at a distant point of the Earth's surface due to a Hertzian oscillator emitting waves of a definite frequency. Previous investigators had obtained approximations† (some of which were incorrect) to the dominant terms of the series which represents the effect due to the Earth, but the earlier approximations cease to be valid in the neighbourhood of the antipodes of the transmitter. On this hypothesis the absolute value of the Hertzian function (with the time-factor suppressed) is roughly proportional to $(\sin\theta)^{-\frac{1}{2}}\exp(-23{\cdot}94\,\lambda^{-\frac{1}{3}}\theta)$, where λ is the wave-length measured in kilometres, and θ is the angular distance from the transmitter. When θ is nearly equal to π, the factor $(\sin\theta)^{-\frac{1}{2}}$ has to be suppressed.

This formula does not agree with results obtained experimentally. The numerical factor 23·94 is much too large, so that, as θ increases, the magnetic force decays much less rapidly than the theory indicates; and it has also been suggested on experimental grounds‡ that the actual state of affairs is represented much more closely when the factor $\lambda^{-\frac{1}{3}}$ is replaced by the factor $\lambda^{-\frac{1}{2}}$.

To secure a considerable modification in the magnitude of the factor 23·94, a suggestion has been put forward by various physicists to the effect that the attenuated gases in the upper atmosphere have an appreciable conductivity, and so they tend to reflect electric waves. This hypothesis was first made in 1900 by Heaviside, and it has been developed more recently by Eccles.§

* 'Roy. Soc. Proc.,' A, vol. 95, pp. 83–99 (1918).

† References to the more important work on the subject are to be found in my previous paper. A complete bibliography is given by Love, 'Phil. Trans.,' A, vol. 215, p. 106 (1915).

‡ See *e.g.*, Austin, 'Washington Bureau of Standards Bulletin' (October, 1911), who suggests $9{\cdot}6\,\lambda^{-\frac{1}{2}}$ in place of $23{\cdot}94\,\lambda^{-\frac{1}{3}}$.

§ Eccles has worked out some of the mathematical consequences of the theory, 'Roy. Soc. Proc.,' A, vol. 87, pp. 79–99 (1912).

It therefore seems desirable to investigate the consequences of the assumption that the Earth is surrounded by a concentric conducting layer at a considerable height. The height which has been suggested by Eccles is from 100 kilom. to 150 kilom. above the surface of the Earth. This comparatively sharply defined region of the atmosphere which plays the predominant part in signalling by night is called "Heaviside's layer" by Eccles. Signals during the day are affected by partial ionisation (due to sunlight) of lower regions of the atmosphere.

A consequence of this theory is that the disturbance produced by the oscillator is confined to the neighbourhood of the Earth; this concentration obviously tends to increase the magnetic force at points on the Earth's surface. And it will, in fact, be shown that, if the Earth be regarded as a perfect conductor surrounded by a concentric perfect reflector, the expression for the Hertzian function consists of a finite number of oscillatory terms combined with an infinite series of negative exponentials.

If the finiteness of the conductivity of the Earth and reflector is taken into account, the character of the finite series is modified; and it appears that the Hertzian function constructed on the reflector theory agrees quite well with the form suggested by experimenters. In particular, it seems impossible to account for the factor $\lambda^{-\frac{1}{2}}$ on any theory of pure diffraction; and the presence of convergent waves is necessary to produce it.

The results of this paper therefore tend to confirm the existence of the reflecting layer. A consequence of its presence is that it places grave obstacles in the way of communications with Mars or Venus, if the desirability of communicating with those planets should ever arise.

2. In this paper the consequences of the ionisation theory are investigated by supposing that the effect produced by the regions of the atmosphere which tend to reflect the waves may be represented by taking the Earth to be surrounded by a concentric reflector of radius c; it is at first supposed that the reflector is perfect, so that the tangential component of electric force is zero, and subsequently the theory is modified by supposing the reflector to have a finite conductivity; this procedure removes the grave analytical difficulties which would arise if the dielectric surrounding the Earth were supposed to have a conductivity which varied continuously. With the exception of the novelty involved in the introduction of c, the notation will be the same as in my previous paper.

The radius of the Earth is denoted by a, the distance of the transmitter from the centre of the Earth is b, and of course $a \leq b < c$.

The transmitter, which is taken to be on the axis of harmonics, emits waves

of period $2\pi/\omega$, and the components* of the electric force and of the magnetic force are (E_r, E_θ, E_ϕ) and (H_r, H_θ, H_ϕ); there are two constants† β, γ of the medium transmitting the waves such that the electric force and the magnetic force are determined in terms of the Hertzian function Π by the equations

$$E_r = -\frac{1}{rb}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\Pi}{\partial\mu}\right\}, \qquad E_\theta = \frac{1}{rb}\frac{\partial^2(r\Pi)}{\partial r\,\partial\theta}, \qquad E_\phi = 0,$$

$$H_r = 0, \qquad H_\theta = 0, \qquad H_\phi = -\frac{\beta}{b}\frac{\partial\Pi}{\partial\theta},$$

where Π satisfies the wave equation

$$(\nabla^2+k^2)\,\Pi = 0,$$

in which $k^2 = -\beta\gamma$.

If the Hertzian function were unaffected by the presence of the Earth and of the reflector, its value would be $\Pi_0 = e^{-ik\mathrm{R}}/\mathrm{R}$, where R is the distance from the transmitter; and Π_0 can be expanded in the forms

$$\Pi_0 = -\frac{i}{krb}\sum_{n=0}^{\infty}(2n+1)\,\zeta_n(kb)\,\psi_n(kr)\,P_n(\mu), \qquad (r<b),$$

$$\Pi_0 = -\frac{i}{krb}\sum_{n=0}^{\infty}(2n+1)\,\zeta_n(kr)\,\psi_n(kb)\,P_n(\mu), \qquad (r>b),$$

where ζ_n and ψ_n are expressible in terms of Bessel functions by the equations

$$\psi_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}\,J_{n+\frac{1}{2}}(x), \qquad \zeta_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}\,H_{n+\frac{1}{2}}(x),$$

the last function being the second of the two Hankel-Nielsen functions, sometimes called Bessel functions of the third kind.

We now take into account the presence of the Earth and of the reflector: the values‡ of β, γ, k in the interior of the Earth are denoted by the symbols β_i, γ_i, k_i; and the disturbance produced in the Hertzian function in the atmosphere is Π_d, while the disturbed function in the interior of the Earth is Π_i The appropriate series for Π_d and Π_i are

$$\Pi_d = -\frac{i}{krb}\sum_{n=0}^{\infty}(2n+1)\left\{a_n\zeta_n(kr)+c_n\psi_n(kr)\right\}P_n(\mu),$$

$$\Pi_i = -\frac{i}{k_i rb}\sum_{n=0}^{\infty}(2n+1)\,b_n\psi_n(k_i r)\,P_n(\mu),$$

where the coefficients a_n, b_n, and c_n are constants.

* The time factor $e^{i\omega t}$ is supposed to be suppressed throughout.

† These constants depend also on the frequency of the waves; their precise values are

$$\beta = \frac{\sigma+i\epsilon\omega}{C}, \qquad \gamma = \frac{i\mu\omega}{C},$$

where σ is the conductivity, ϵ is the dielectric constant, μ is the permeability, and C is the velocity of light; the symbol μ is used elsewhere to denote $\cos\theta$, where θ is the angular distance from the transmitter, but this will cause no confusion.

‡ It should be noted that k is positive in the case of air, but it is a complex number in the interior of the Earth.

The boundary conditions which express the continuity of the tangential components of electric force and magnetic force, when the reflector is taken to be a perfect conductor, are effectively

$$\beta(\Pi_0+\Pi_d) = \beta_i\Pi_i, \qquad (r = a),$$

$$\frac{\partial}{\partial r}(r\Pi_0+r\Pi_d) = \frac{\partial}{\partial r}(r\Pi_i), \qquad (r = a),$$

$$\frac{\partial}{\partial r}(r\Pi_0+r\Pi_d) = 0. \qquad (r = c).$$

These conditions yield the following system of equations to determine a_n, b_n and c_n:—

$$(\beta/k)\left[a_n\zeta_n(ka)+\{\zeta_n(kb)+c_n\}\psi_n(ka)\right] = (\beta_i/k_i)\,b_n\psi_n(k_ia),$$

$$a_n\zeta_n'(ka)+\{\zeta_n(kb)+c_n\}\psi_n'(ka) = b_n\psi_n'(k_ia),$$

$$\{a_n+\psi_n(kb)\}\zeta_n'(kc)+c_n\psi_n'(kc) = 0.$$

On solving we find that

$$a_n = -\begin{vmatrix} \zeta_n(kb)\psi_n(ka) & \psi_n(k_ia) & \psi_n(ka) \\ \zeta_n(kb)\psi_n'(ka) & \beta k_i\psi_n'(k_ia)/(\beta_ik) & \psi_n'(ka) \\ \psi_n(kb)\zeta_n'(kc) & 0 & \psi_n'(kc) \end{vmatrix} \div\Delta_n,$$

$$c_n = -\begin{vmatrix} \zeta_n(ka) & \psi_n(k_ia) & \zeta_n(kb)\psi_n(ka) \\ \zeta_n'(ka) & \beta k_i\psi_n'(k_ia)/(\beta_ik) & \zeta_n(kb)\psi_n'(ka) \\ \zeta_n'(kc) & 0 & \psi_n(kb)\zeta_n'(kc) \end{vmatrix} \div\Delta_n,$$

$$\text{where } \Delta_n = \begin{vmatrix} \zeta_n(ka) & \psi_n(k_ia) & \psi_n(ka) \\ \zeta_n'(ka) & \beta k_i\psi_n'(k_ia)/(\beta_ik) & \psi_n'(ka) \\ \zeta_n'(kc) & 0 & \psi_n'(kc) \end{vmatrix}$$

and hence the value of the Hertzian function just outside the surface of the Earth can be reduced to

$$\Pi_b(a,\theta) = -\frac{1}{kab}\sum_{n=0}^{\infty}(2n+1)\,P_n(\mu)\,\psi_n(k_ia)\left[\psi_n(kb)\,\zeta_n'(kc) - \psi_n'(kc)\,\zeta_n(kb)\right]\div\Delta_n.$$

It should be noticed that this expression does not reduce to the expression given in my former paper when c is made infinite; the reason for this is that the presence of a reflector at infinity would produce convergent waves while the Hertzian function of the previous paper contained only divergent waves outside the sphere $r = b$.

3. Now that the general form of the Hertzian function has been obtained, we must apply a transformation of the type introduced in my previous paper in order to render the function susceptible to numerical computation.

With this object in view, it is desirable to obtain the corresponding formula

for the special case in which the Earth is supposed to be a perfect conductor, and then to modify the solution by taking account of the fact that $\beta k_i/(\beta_i k)$ is not zero.

4. In the case of a perfect conductor, the Hertzian function is

$$\Pi_b^{(0)}(a,\theta) = -\frac{1}{kab}\sum_{n=0}^{\infty}(2n+1)\,\mathrm{P}_n(\mu)\frac{\psi_n(kb)\,\zeta_n'(kc)-\psi_n'(kc)\,\zeta_n(kb)}{\psi_n'(ka)\,\zeta_n'(kc)-\psi_n'(kc)\,\zeta_n'(ka)};$$

and it is to be noted that resonance occurs when any of the denominators vanish. As it will be found that resonance effects do not occur in the actual physical problem (though they may occur in the ideal problem now under consideration), it is convenient to disregard them.

The series which has just been obtained for $\Pi_b^{(0)}(a,\theta)$ is the sum of the residues of

$$-\frac{2s\pi}{kab}\cdot\frac{\mathrm{P}_{s-\frac{1}{2}}(-\mu)}{\cos s\pi}\,\frac{\{\psi_{s-\frac{1}{2}}(kb)\,\zeta_{s-\frac{1}{2}}'(kc)-\psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(kb)\}}{\{\psi_{s-\frac{1}{2}}'(ka)\,\zeta_{s-\frac{1}{2}}'(kc)-\psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}'(ka)\}}$$

at the poles $s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$; and since it is assumed that there are no resonance effects, these are all simple poles.

Now those poles of the function which are not zeros of $\cos s\pi$ are zeros of

$$\psi_{s-\frac{1}{2}}'(ka)\,\zeta_{s-\frac{1}{2}}'(kc)-\psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}'(ka);$$

and it will appear in § 5 that the only zeros of this function of s lie on the cruciform figure,* which consists of the imaginary axis in the s-plane together with that part of the real axis which joins the points $s=\pm kc$.

We now consider the integral

$$\frac{2\pi}{kab}\int\frac{s\mathrm{P}_{s-\frac{1}{2}}(-\mu)}{\cos s\pi}\cdot\frac{\{\psi_{s-\frac{1}{2}}(kb)\,\zeta_{s-\frac{1}{2}}'(kc)-\psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(kb)\}}{\{\psi_{s-\frac{1}{2}}'(ka)\,\zeta_{s-\frac{1}{2}}'(kc)-\psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}'(ka)\}}\,ds,$$

taken round a contour which consists of a semicircle of large radius, R, with centre at the origin, lying on the right of the imaginary axis, together with that part of the imaginary axis which joins the points $\pm\mathrm{R}i$; on the latter part of the contour, Cauchy's principal value has to be taken, in view of the poles on the imaginary axis. It is readily verified that the integrand is an odd function of s, and so the integral along the imaginary axis is equal to πi times the sum of the residues at the poles on it, the rest of the integral cancelling. On making R tend to infinity through such values that the semicircle never passes through a pole, we find (exactly as in my previous paper) that the integral round the semicircle tends to zero. It follows that the sum of the residues of the integrand at all the poles on the positive half of the real axis plus half the sum of the residues at the poles on the imaginary axis

* The change which the presence of the reflector produces in the topography of the poles is very remarkable.

is zero; the integrand being an odd function of s, the latter term may be replaced by the sum of the residues at the poles on the negative half of the imaginary axis.

For brevity, we shall write x, X, y respectively for a, b, c; and we shall denote those zeros of

$$\psi_{s-\frac{1}{2}}'(x)\,\zeta_{s-\frac{1}{2}}'(y)-\psi_{s-\frac{1}{2}}'(y)\,\zeta_{s-\frac{1}{2}}'(x),$$

which are under consideration by $\nu_1, \nu_2, \nu_3, \ldots$, so that either $\mathrm{R}(\nu)\geq 0$ and $\mathrm{I}(\nu)=0$, or else $\mathrm{R}(\nu)=0$ and $\mathrm{I}(\nu)\leq 0$; and we have

$$\Pi_b^{(0)}(a,\theta)=\frac{2\pi}{kab}\sum_\nu\frac{\nu\mathrm{P}_{\nu-\frac{1}{2}}(-\mu)}{\cos\nu\pi}\,\frac{\{\psi_{\nu-\frac{1}{2}}(\mathrm{X})\,\zeta_{\nu-\frac{1}{2}}'(y)-\psi_{\nu-\frac{1}{2}}'(y)\,\zeta_{\nu-\frac{1}{2}}(\mathrm{X})\}}{[\partial\{\psi_{s-\frac{1}{2}}'(x)\,\zeta_{s-\frac{1}{2}}'(y)-\psi_{s-\frac{1}{2}}'(y)\,\zeta_{s-\frac{1}{2}}'(x)\}/\partial s]_{s=\nu}}.$$

5. We shall next prove the theorem which has just been assumed, namely, that the number of real values of ν is finite, that the number of purely imaginary values is infinite, and that there are no other complex values of ν. It will also be shown that the imaginary values of ν are such that the series just obtained for $\Pi_b^{(0)}(a,\theta)$ converges very rapidly except when θ is quite small; and that the convergence is uniform with respect to θ and X when $\theta_0\leq\theta\leq\pi$ and $y>\mathrm{X}\geq x$, where θ_0 is an arbitrary small positive angle. It follows from these results that term-by-term differentiation with respect to θ is permissible, and also (by Abel's theorem on continuity) that x may be written in place of X to obtain the effect due to an oscillator placed on the surface of the Earth.

The only inconvenient property of the series is the number of real values of ν; there may be as many as 60 or 80 such values. The corresponding terms of the series are of an oscillatory character, and in this ideal problem numerical computation would be somewhat tedious.

We now define the function $\mathrm{F}(s)$ by the equation

$$\begin{aligned}\mathrm{F}(s)&=\psi_{s-\frac{1}{2}}'(x)\,\zeta_{s-\frac{1}{2}}'(y)-\psi_{s-\frac{1}{2}}'(y)\,\zeta_{s-\frac{1}{2}}'(x)\\&=\tfrac{1}{4}\pi\frac{\partial^2}{\partial x\partial y}[(xy)^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,\mathrm{H}_s^{(2)}(y)-\mathrm{H}_s^{(1)}(y)\,\mathrm{H}_s^{(2)}(x)\}],\end{aligned}$$

and we proceed to investigate the zeros of $\mathrm{F}(s)$.

As in my previous paper, we write $s=x\cosh u$, and we shall also write $s=y\cosh v$; further, we define $\mathrm{S}_s^{(1)}(x)$ and $\mathrm{S}_s^{(2)}(x)$ to be the (discontinuous) solutions of Bessel's equation, whose asymptotic expansions (of Debye's type) when s and x are large but not nearly equal, are given by the formulæ

$$\mathrm{S}_s^{(1)}(x)\sim e^{(\sinh u-u\cosh u)-\frac{1}{4}\pi i}\div\sqrt{\{\tfrac{1}{2}\pi x\sin(-iu)\}},$$

$$\mathrm{S}_s^{(2)}(x)\sim e^{-x(\sinh u-u\cosh u)+\frac{1}{4}\pi i}\div\sqrt{\{\tfrac{1}{2}\pi x\sin(-iu)\}}.$$

It follows from the Tables given in my previous paper (p. 93) that

$$F(s) = \tfrac{1}{4}\pi \frac{\partial}{\partial x \partial y} [(xy)^{\frac{1}{2}} \{S_s^{(1)}(x) S_s^{(2)}(y) - S_s^{(1)}(y) S_s^{(2)}(x)\}],$$

except in narrow bands of the plane, in which the Hankel-Neilsen functions of x and y have asymptotic expansions of different types, the function of x involving two functions of the type S, while the function of y involves only one such function. In these bands the expression just given for $F(s)$ has to be modified by the insertion of terms which are negligible in comparison with the terms already present, unless s is nearly equal to $\pm x$ or $\pm y$.

Hence the only zeros of $F(s)$ in the whole plane are near the curves on which the real part of

$$x(\sinh u - u \cosh u) - y(\sinh v - v \cosh v)$$

vanishes, or else they are near the points $\pm x$, $\pm y$.

Regarding x and y as fixed (s, of course, being a complex variable), we write

$$\chi(s) = x(\sinh u - u \cosh u) - y(\sinh v - v \cosh v).$$

It is found that $\chi'(s) = v - u$ and that $R\chi(s)$ vanishes on the cruciform figure consisting of the imaginary axis and of that part of the real axis which joins the points $s = \pm x$.

Also, when $|s|$ is large compared with x and y,

$$-\chi(s) = s \log(y/x) - \tfrac{1}{4}(y^2 - x^2)/s - \ldots.$$

It is now easy to prove that $R\chi(s)$ does not vanish except on the cruciform figure. For if there were any curve on which $R\chi(s)$ vanished, this curve would either (i) be closed, or (ii) have points at infinity, or (iii) terminate on the cruciform figure, or (iv) terminate on the cuts from $\pm x$ to $\pm\infty$.

These possibilities do not in fact occur, for (i) the closed curve would contain a zero of $\chi'(s)$, and there is no such point except $s = 0$; (ii) $R\chi(s)$ vanishes at infinity only on the imaginary axis; (iii) the end points would be zeros of $\chi'(s)$, and no curves on which $R\chi(s)$ vanishes emerge from the origin except the axes; (iv) $R\chi(s)$ is negative on the cut from $+x$ to $+\infty$, and it is positive on the other cut.

We shall next determine the number of zeros of $F(s)$ in an arbitrarily large region of the plane, and we shall show that this number is the same as the number of zeros of $F(s)$, which lie on the cruciform figure formed by the imaginary axis and that part of the real axis which joins the points $\pm y$; and it is then obvious that all the zeros of $F(s)$ lie on this cruciform figure.

Consider $\int \{d \log F(s)/ds\} ds$ taken round a rectangle in the s-plane with

corners at the points $\pm M \pm (N+\frac{1}{2})\pi i/\log(y/x)$, when N is a large integer. When we make $M \to \infty$, the integrals along the vertical sides of the rectangle tend to zero.

Since N is large, we may replace the functions $S_s^{(1)}$, $S_s^{(2)}$ by the dominant terms in their asymptotic expansions, and so the change of phase of $F(s)$ when s traverses a horizontal side of the rectangle is the same as the change of phase of $\sinh\chi(s)$, hence the total change of phase of $F(s)$ as s describes the rectangle as $4N\pi$. The total number of zeros inside the rectangle is therefore 2N.

Next consider the zeros of $F(s)$ which lie on the cruciform figure formed by the imaginary axis and the part of the real axis joining $\pm x$; on this curve $\chi(s)$ is a pure imaginary. Also, by using Debye's integrals, it may be shown* that, with each value of s, which makes $i\chi(s)/\pi$ equal to an integer, is associated one and only one zero of $F(s)$, and this zero is on the cruciform figure; the difference between the value of s, which makes $i\chi(s)/\pi$ an integer, and the corresponding zero of $F(s)$ may be calculated by Stokes' method when u is not small.

Now $$i\chi(x)/\pi = \{y^2-x^2\}^{\frac{1}{2}} - y\cos^{-1}(x/y)\}/\pi;$$

we take N_1 to be the greatest integer contained in this number, and write $N = N_1+N_2$. Then, as s passes along the real axis from x to 0, and then along the imaginary axis to $(N+\frac{1}{2})\pi i/\log(y/x)$, $i\chi(s)/\pi$ increases from $i\chi(x)/\pi$ to (approximately) $N+\frac{1}{2}$, and so s passes through N_2 zeros of $F(s)$.

To show that $F(s)$ has N_1 zeros between $s = x$ and $s = y$, we express $F(s)$ in terms of Bessel functions of the first and second kinds by the formula

$$F(s) = \tfrac{1}{2}\pi i \frac{\partial^2}{\partial x \partial y}\left[(xy)^{\frac{1}{2}}\{\Upsilon_s(x)J_s(y) - \Upsilon_s(y)J_s(x)\}\right].$$

Now as s decreases from y to x, the function $\Theta_s(x)$, defined by the equation

$$\Theta_s(x) = \frac{d\{\Upsilon_s(x)\sqrt{x}\}}{dx}\bigg/\frac{d\{J_s(x)\sqrt{x}\}}{dx},$$

remains bounded,† since the order of the Bessel function exceeds its argument; but $\Theta_s(y)$ passes through an infinity for each zero of its denominator, and there is one such zero for each multiple of πi, which $y(\sinh v - v\cosh v)$ passes through; since this function varies from 0 to $i\chi(x)$ through purely imaginary values, there are N_1 such zeros.

Also $\Theta_s(y)$ is a decreasing function of s, and it may be shown (by using Bessel's equation) that $\Theta_y(\xi)$ is an increasing function of ξ when $\xi \leq y$, so

* The proof is substantially identical with that given in the simpler case of a single Bessel function (of the first or second kind), 'Roy. Soc. Proc.,' A, vol. 94, p. 193 (1918).

† 'Proc. Lond. Math. Soc.' (2), vol. 16, p. 166 (1917).

that $\Theta_y(x) < \Theta_y(y)$. Consequently the graphs of $\Theta_s(x)$ and $\Theta_s(y)$, *qua* functions of s, intersect exactly N_1 times when s lies between x and y, and each intersection is just on the left of the corresponding value of s which makes $y(\sinh v - v\cosh v)$ a multiple of πi.

Hence there are N_1 zeros of $F(s)$ between $s = x$ and $s = y$; and since $F(s)$ is an odd function there are evidently $2(N_1+N_2)$ zeros in all on the (large) cruciform figure.

Hence all the zeros of $F(s)$ are on the cruciform figure; and this is the result which had to be established.

The number of positive zeros of $F(s)$ is the integer next less than $i\chi(0)/\pi$. Since $i\chi(0) = y-x$, we can easily construct the following Tables. In these Tables h denotes the height of the reflector above the surface of the Earth, and λ denotes the wave-length, both being measured in kilometres.

$\lambda = 5.$

h	100	150	200
$i\chi(0)/\pi$	40	60	80

$\lambda = 10.$

h	100	150	200
$i\chi(0)/\pi$	20	30	40

6. We now consider the value of

$$\{\psi_{\nu-\frac{1}{2}}(X)\,\zeta_{\nu-\frac{1}{2}}'(y) - \psi_{\nu-\frac{1}{2}}'(y)\,\zeta_{\nu-\frac{1}{2}}(X)\}/F'(\nu).$$

When x, y, and X are regarded as fixed, while ν tends to infinity through its purely imaginary values, it is easy to show that this function is bounded, and so the series for the Hertzian function converges like $\Sigma\nu\sec\nu\pi\,.\,P_{\nu-\frac{1}{2}}(-\cos\theta)$; and in view of the rapid convergence of this series (except near $\theta = 0$) it follows that the series for the Hertzian function converges uniformly when $x \leq X \leq y$.

The statements made at the beginning of § 5 are therefore established. We shall consequently suppose that the transmitter is actually on the surface of the Earth; the error produced by neglecting the height of the transmitter is minute.

Furthermore, if we use the approximations

$$\frac{\partial S_s^{(1)}(x)}{\partial s} \sim -uS_s^{(1)}(x), \qquad \frac{\partial S_s^{(2)}(x)}{\partial s} \sim uS_s^{(2)}(x),$$

$$\frac{\partial S_s^{(1)}(x)}{\partial x} \sim \sinh u\,.\,S_s^{(1)}(x), \qquad \frac{\partial S_s^{(2)}(x)}{\partial x} \sim -\sinh u\,.\,S_s^{(2)}(x),$$

which are valid when S is not nearly equal to x, we find that

$$\psi_{\nu-\frac{1}{2}}(x)\,\zeta_{\nu-\frac{1}{2}}'(y) - \psi_{\nu-\frac{1}{2}}'(y)\,\zeta_{\nu-\frac{1}{2}}(x)\}/F'(\nu) \sim 1/[(v-u)\sinh u]_{s=\nu},$$

and this is greatest when $v-u$ or $\sinh u$ is very small, *i.e.*, when ν is nearly

equal to 0 or x. The residue is not really infinite when ν is 0 or x, because closer approximations have to be used in calculating it.

Hence $\Pi_a^{(0)}(a, \theta)$ is expressed as a series which consists of a finite number of terms which are oscillatory functions of θ combined with an infinite number of terms each of which is roughly proportional to an expression of the type $\sqrt{\operatorname{cosec}\theta} . \exp(-|\nu|\theta)$.

It seems unnecessary to examine this series more closely; our next object must be to consider modifications produced in the oscillatory terms by taking into account the imperfect conductivity of the Earth.

7. With the aid of the results contained in §§ 4–6 we shall examine the problem in which it is supposed that the Earth consists of an imperfect conductor surrounded by a perfect reflector in the upper regions of the atmosphere.

The contour integral to be considered is now

$$\frac{2\pi}{kab}\int \frac{s\mathrm{P}_{s-\frac{1}{2}}(-\mu)}{\cos s\pi} \frac{\psi_{s-\frac{1}{2}}(kb)\,\zeta_{s-\frac{1}{2}}'(kc) - \psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(kb)}{\mathrm{G}(s)}\,ds,$$

where

$$\mathrm{G}(s) = \psi_{s-\frac{1}{2}}'(ka)\,\zeta_{s-\frac{1}{2}}'(kc) - \psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}'(ka)$$
$$-\frac{\beta k_i}{\beta_i k}\frac{\psi_{s-\frac{1}{2}}'(k_i a)}{\psi_{s-\frac{1}{2}}(k_i a)}\{\psi_{s-\frac{1}{2}}(ka)\,\zeta_{s-\frac{1}{2}}'(kc) - \psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(ka)\}.$$

Now in the case of signals over a sheet of water the high conductivity of the water ensures that the disturbance at any depth is negligible, and so the solution is the same as if the Earth consisted of a sphere of water; and in the case of signals over dry land the part of the disturbance which does not travel more or less directly along a geodesic from the receiver to the transmitter, produces little effect on the transmitter and so the solution is the same as if the Earth consisted of a sphere whose conductivity is that of dry earth at the surface of the Earth.

Now $$k_i^2/k^2 = -i(\sigma_i/\omega) + \epsilon_i,$$

where ω is the frequency of the waves and the values of σ_i, ϵ_i, and σ_i/ω are given in the following Table:—

	σ_i.	ϵ_i.	σ_i/ω.
Earth	10^7	4	$5 \cdot 27\,\lambda$
Water	$4 \cdot 26 \times 10^{11}$	81	$225{,}000\,\lambda$

It follows that, in both cases, $|k_i a|$ is large compared with the real values of ν, and the phase of $k_i a$ is nearly equal to $-\frac{1}{4}\pi$; also

$$\beta k_i/(\beta_i k) = k/k_i,$$

and this is quite small. Hence the zeros of G (s) are associated with zeros of F (s), and in calculating the zero of G (s) which is associated with any particular real zero of F (s), we may take

$$\psi_{s-\frac{1}{2}}'(k_i a)/\psi_{s-\frac{1}{2}}(k_i a) = i.$$

Let $\nu+\vartheta$ be the zero of G (s) which is associated with the zero ν of F (s). In estimating the value of ϑ, we have to use different approximate formulæ for the various Bessel functions according as ν is or is not remote from x and y.

In the easy case when ν is not near x we have approximately

$$\chi(s+\vartheta) = \chi(s)+(v-u)\vartheta,$$

and hence ϑ is determined by the approximate equation

$$\sinh u \sinh\{\chi(\nu)+(v-u)\vartheta\} = (ik/k_i)\cosh\{\chi(\nu)+(v-u)\vartheta\},$$

where $\sinh\chi(\nu) = 0$. Hence

$$\vartheta = \frac{1}{2(v-u)}\log\frac{\sinh u+ik/k_i}{\sinh u-ik/k_i}.$$

Now $v-u$ is a pure imaginary when ν lies between 0 and x; and so I (ϑ), which is our essential requirement, is given by the formula

$$\mathrm{I}(\vartheta) = -\frac{1}{2|v-u|}\log\left|\frac{\sinh u+ik/k_i}{\sinh u-ik/k_i}\right|.$$

Except near the critical points $0, x, y$, it may be shown that $(v-u)\sinh u$ is the change produced in $\cosh u$ when x is replaced by y. Hence

$$(v-u)\sinh u = (\nu/y)-(\nu/x) = -(h/a)\cosh u,$$

where h is the height of the reflector. It follows that

$$\mathrm{I}(\vartheta) = -\frac{a}{2h}\left|\tanh u\right| . \log\left|\frac{\sinh u+ik/k_i}{\sinh u-ik/k_i}\right|.$$

For simplicity of calculation we shall take the phase of k/k_i to be exactly $\frac{1}{4}\pi$, and then we shall write

$$u = i\phi, \qquad k = k_i\,\delta\exp\tfrac{1}{4}\pi i.$$

We deduce that

$$\mathrm{I}(\vartheta) = -\frac{a}{4h}\tan\phi\log\frac{\delta^2+\delta\sqrt{2}\sin\phi+\sin^2\phi}{\delta^2-\delta\sqrt{2}\sin\phi+\sin^2\phi}.$$

The expression on the left increases steadily from* $-\infty$ to 0 as ϕ decreases from $\frac{1}{2}\pi$ to 0, but its rate of change is, on the whole, quite slow. Thus, so long as $\sin\phi$ is fairly small (but large compared with δ), the expression is

* When $\nu = 0$, I(ϑ) is not actually infinite, because our approximation then becomes inadequate ; and it turns out that in reality I(ϑ) is only moderately large.

nearly equal to $-a\delta/h\surd 2)$; and when $\sin\phi = \delta$ it is nearly equal to $-0{\cdot}44\, a\delta/h$. In the former case we have approximately

$$\vartheta = \frac{ik/k_i}{(v-u)\sinh u} = -aik/(hk_i);$$

so the phase of ϑ is $-\frac{1}{4}\pi$, approximately.

In the case of values of ν which are nearly equal to x or y, we have to use the appropriate expansions* for Bessel functions of large order. As a moderately simple case which can be worked out in detail, we shall consider what happens if a value of ν is actually equal to x.

We then have

$$\begin{aligned} \mathrm{F}(x) &= \tfrac{1}{4}\pi\left[\frac{\partial^2}{\partial x\partial y}(xy)^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,\mathrm{H}_s^{(2)}(y)-\mathrm{H}_s^{(1)}(y)\,\mathrm{H}_s^{(2)}(x)\}\right]_{s=x}, \\ &= 0 \\ \mathrm{G}(x+\vartheta) &= \tfrac{1}{4}\pi\left[\frac{\partial^2}{\partial x\partial y}(xy)^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,\mathrm{H}_s^{(2)}(y)-\mathrm{H}_s^{(1)}(y)\,\mathrm{H}_s^{(2)}(x)\}\right]_{s=x+\vartheta} \\ &\quad -\tfrac{1}{4}\pi(ik/k_i)\left[\frac{\partial}{\partial y}(xy)^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,\mathrm{H}_s^{(2)}(y)-\mathrm{H}_s^{(1)}(y)\,\mathrm{H}_s^{(2)}(x)\right]_{s=x+\vartheta} \\ &= 0. \end{aligned}$$

Also $\qquad \mathrm{H}_{x+\vartheta}^{(1)}(y) = e^{-v\vartheta}\mathrm{H}_x^{(1)}(y), \qquad \mathrm{H}_{x+\vartheta}^{(2)}(y) = e^{v\vartheta}\mathrm{H}_x^{(2)}(y),$

where v has the value corresponding to $s = x$, so that $v = i\surd(2h/a)$.

On using these results to simplify the equation $\mathrm{G}(x+\vartheta) = 0$, and then eliminating the Bessel functions involving y by means of the equation $\mathrm{F}(x) = 0$, and making the further simplification produced by the approximate equation

$$\left[\frac{\partial}{\partial x}\{x^{\frac{1}{2}}\,\mathrm{H}_s^{(1)}(x)\}\right]_{s=x} = e^{\frac{2}{3}\pi i}\left[\frac{\partial}{\partial x}\{x^{\frac{1}{2}}\mathrm{H}_s^{(2)}(x)\}\right]_{s=x}$$

(which follows at once from a formula due to Debye), we find that

$$\frac{\partial}{\partial x}[x^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,e^{v\vartheta}-e^{\frac{2}{3}\pi i}\mathrm{H}_s^{(2)}(x)\,e^{-v\vartheta}\}]_{s=x+\vartheta}$$

$$-(ik/k_i)\,[x^{\frac{1}{2}}\{\mathrm{H}_s^{(1)}(x)\,e^{v\vartheta}-e^{\frac{2}{3}\pi i}\mathrm{H}_s^{(2)}(x)\,e^{-v\vartheta}\}]_{s=x+\vartheta} = 0.$$

If we put $\xi = \frac{1}{3}x^{-\frac{1}{2}}(-2\vartheta)^{\frac{3}{2}}$ and substitute for the Bessel functions their approximate values in terms of Bessel functions of orders $\frac{1}{3}$ and $-\frac{2}{3}$, this equation may be written

$$(3\xi/x)^{\frac{1}{3}}\,[\mathrm{H}_{-\frac{2}{3}}^{(1)}(\xi)\,e^{v\vartheta}-e^{\frac{1}{3}\pi i}\mathrm{H}_{-\frac{2}{3}}^{(2)}(\xi)\,e^{-v\vartheta}]$$

$$-(ik/k_i)\,[\mathrm{H}_{\frac{1}{3}}^{(1)}(\xi)\,e^{v\vartheta}-e^{\frac{1}{3}\pi i}\mathrm{H}_{\frac{1}{3}}^{(2)}(\xi)\,e^{-v\vartheta}] = 0;$$

that is to say,

$$(3\xi/x)^{\frac{1}{3}}\,[\mathrm{J}_{-\frac{2}{3}}(\xi)\sinh v\vartheta+\mathrm{J}_{\frac{2}{3}}(\xi)\sinh(v\vartheta+\tfrac{1}{3}\pi i)]$$

$$+(ik/k_i)\,[\mathrm{J}_{-\frac{1}{3}}(\xi)\sinh(v\vartheta+\tfrac{1}{3}\pi i)-\mathrm{J}_{\frac{1}{3}}(\xi)\sinh v\vartheta] = 0.$$

* 'Proc. Camb. Phil. Soc.,' vol. 19, p. 110 (1917); 'Proc. Roy. Soc.,' A, vol. 95, p. 96 (1918).

Now if the value of ϑ given by this equation is not appreciably larger* than the value previously found, ξ does not exceed about $\frac{1}{4}$ (when the wave-length is short), and it is less with a greater wave-length. Hence, in approximating to the Bessel functions, we may safely neglect all except the first terms of the functions of negative order, so that

$$\frac{\sinh v\vartheta}{\sinh(v\vartheta+\frac{1}{3}\pi i)} = -(ik/k_i).(\tfrac{1}{6}x)^{\frac{1}{3}}\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{2}{3}) = \eta,$$

say; and, to a sufficient approximation,

$$v\vartheta = i\eta\sqrt{3}/(2-\eta),$$

where $\qquad v = i\sqrt{(2h/a)} \qquad$ and $\qquad |2-\eta| < 2,$

since $\arg\eta = -\frac{1}{4}\pi$.

Hence the modulus of ϑ is greater than if ϑ were given by the formula

$$v\vartheta = i\eta\sqrt{(\tfrac{3}{4})},$$

and the phase of ϑ is less than $-\frac{1}{4}\pi$.

Now the imaginary part of ϑ given by this formula is numerically greater than that given by the formula obtained when ν is not nearly equal to x if

$$\tfrac{1}{2}\sqrt{(\tfrac{3}{2}a/h)}.(\tfrac{1}{6}x)^{\frac{1}{3}}\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{2}{3}) > 0{\cdot}63\,(a/h),$$

i.e., if $\qquad h > 4{\cdot}9\,\lambda^{\frac{2}{3}}.$

Now the greatest value of wave-lengths used in practice is 10 kilom., and, since it is certain that $h > 23$, the zeros of $G(s)$, which are nearly equal to x, are less important than those whose real part is appreciably less than x.

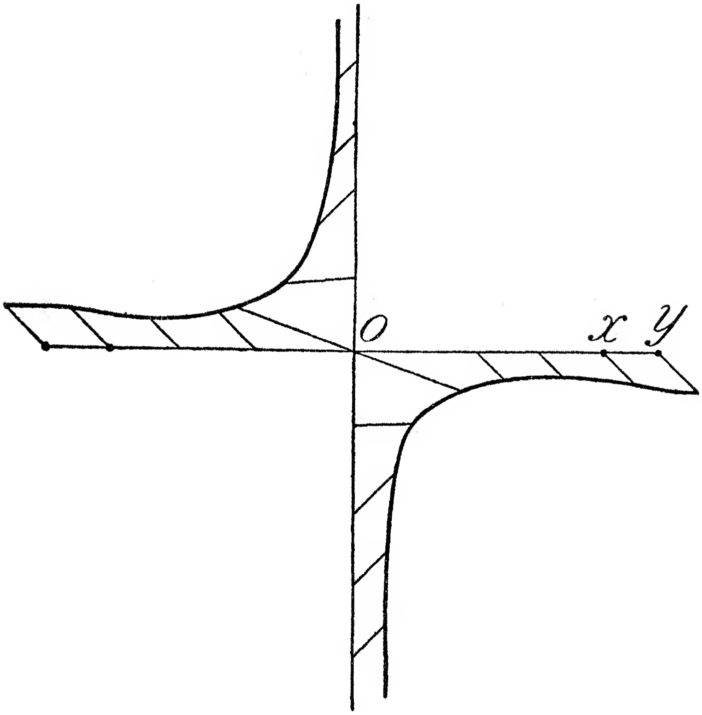
The zeros of $G(s)$, which are nearly equal to y, may be treated in a similar manner, with the same result.

Now, in the case of waves over dry land, δ is about as small a value as may be given to $\sin\phi$ when the simple approximation is used; in the case of waves over water, the smallest admissible value of $\sin\phi$ is much greater than δ.

Hence we may infer that, in the case of waves over dry land, the bulk of the zeros of $G(s)$, which are in the neighbourhood of the real axis, are between the lines on which $I(\vartheta)$ has the values $-0{\cdot}7\,a\delta/h$ and $-0{\cdot}44\,a\delta/h$, none being nearer than the latter line; in the case of waves over the sea, they are clustered much more closely about the former line, except the zeros near the point x; these zeros are nearer the real axis when the wave-lengths are long (but they are not nearer in the case of the comparatively short wave-length of 3·75 kilom. used by Austin; for a wave-length of 10 kilom., they are not nearer if $h > 60$, as is probably the case).

* If the value were appreciably larger, the contribution to the Hertzian function of the corresponding term would be smaller.

The values of ϑ corresponding to different values of ν are indicated roughly by the figure.



It should be observed that no zero of $G(s)$ can lie on the positive half of the real axis; for, if it could, then either (i) $k_i^{-1}\psi_{s-\frac{1}{2}}'(k_i a)/\psi_{s-\frac{1}{2}}(k_i a)$ would be real with s positive and k_i complex, and this is not the case; or else (ii) both

$$\psi_{s-\frac{1}{2}}(x)\,\zeta_{s-\frac{1}{2}}'(y)-\psi_{s-\frac{1}{2}}'(y)\,\zeta_{s-\frac{1}{2}}(x)$$

and its derivate with respect to x would vanish simultaneously; this would involve $\zeta_{s-\frac{1}{2}}'(y)$ and $\psi_{s-\frac{1}{2}}'(y)$, having a common zero, which is not the case, or else the Wronskian of $\psi_{s-\frac{1}{2}}(x)$ and $\zeta_{s-\frac{1}{2}}(x)$ would vanish, which is impossible, since its value is $-i$.

8. If, now, we take the integral introduced at the beginning of § 7, and evaluate it in the usual manner, we find that the transformed Hertzian function consists of three parts. The first part is a finite series of not more than 80 terms, corresponding to the real zeros of $F(s)$; in this series the terms are of the order of magnitude of $P_{\nu-\frac{1}{2}}(-\cos\theta)\sec.\nu\pi$, where $I(\nu)$ lies between $-0{\cdot}7\,a\delta/h$ and $-0{\cdot}44\,a\delta/h$; these terms may be taken as equivalent to $\sqrt{\operatorname{cosec}\theta}\,.\exp(-\frac{1}{2}a\delta\theta/h)$ in absolute magnitude, except when θ is nearly equal to π.

The second part is a rapidly convergent infinite series, in which the terms are of the same type as in the first series, but ν tends to infinity in the direction of the imaginary axis, so that the Legendre functions may be replaced by negative exponentials. This series is negligible compared with the first series.*

The third part is the integral

$$\frac{2\pi}{kab}\int_{-\infty i}^{\infty i} \frac{s\mathrm{P}_{s-\frac{1}{2}}(-\mu)}{\cos s\pi} \frac{\{\psi_{s-\frac{1}{2}}(kb)\,\xi_{s-\frac{1}{2}}'(kc) - \psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(kb)\}}{\mathrm{G}(s)}\,ds$$

$$= \frac{2\pi}{kab}\int_{0}^{\infty i} \frac{s\mathrm{P}_{s-\frac{1}{2}}(-\mu)}{\cos s\pi} \{\psi_{s-\frac{1}{2}}(kb)\,\zeta_{s-\frac{1}{2}}'(kc) - \psi_{s-\frac{1}{2}}'(kc)\,\zeta_{s-\frac{1}{2}}(kb)\} \left[\frac{1}{\mathrm{G}(s)} - \frac{1}{\mathrm{G}(-s)}\right] ds.$$

Now $1/\mathrm{G}(s) - 1/\mathrm{G}(-s)$ contains as a factor

$$\frac{\psi_{s-\frac{1}{2}}'(k_i a)}{\psi_{s-\frac{1}{2}}(k_i a)} - \frac{\psi_{-s-\frac{1}{2}}'(k_i a)}{\psi_{-s-\frac{1}{2}}(k_i a)},$$

and, as in my former paper, the presence of this factor makes the integral (or its principal value) negligible in comparison with the first of the two series.

Hence the order of magnitude of the Hertzian function is

$$\sqrt{\operatorname{cosec}\theta}\,.\,\exp(-\tfrac{1}{2}a\delta\theta/h),$$

except near the antipodes of the oscillator.†

Now
$$\tfrac{1}{2}a\delta/h = \tfrac{1}{2}a(\sigma_i/\omega)^{-\frac{1}{2}}/h$$
$$= \frac{10^4}{\pi h}\left(\frac{2\pi\mathrm{C}}{\sigma_i\lambda}\right)^{\frac{1}{2}}.$$

In the case of waves over dry land, this is $1400/(h\sqrt{\lambda})$; in the case of waves over the sea, it is $7/(h\sqrt{\lambda})$.

The latter expression is evidently excessively small compared with the value suggested by Austin, $9{\cdot}6/\sqrt{\lambda}$. The inference is that the upper regions of the atmosphere cannot be treated as a perfect conductor.

We shall therefore have to attack the problem of an imperfectly conducting sphere surrounded by a concentric shell of imperfectly conducting material, the region between the sphere and the shell being filled with homogeneous dielectric.‡

* There is no sharp dividing line between the end of the first series and the beginning of the second.

† Near the antipodes the factor $\sqrt{\operatorname{cosec}\theta}$ has to be suppressed.

‡ The investigation of the terms of the Hertzian function corresponding to zeros of $\mathrm{G}(s)$ near $s = 0$, and between $s = x$ and $s = y$, has been suppressed deliberately, in order that the paper should not be overloaded with analysis.

9. In the problem which has now to be considered, we suppose that the constants in the upper atmosphere have the values β_e, γ_e, κ_e; since the upper atmosphere is a good conductor, the Hertzian function is inappreciable when r exceeds c by a comparatively small amount, so that the error produced by supposing that the conductor extends to infinity is negligible.

We take Π_d and Π_i to have the same forms as in § 2, and we suppose that the Hertzian function in the conductor is Π_e where

$$\Pi_e = -\frac{i}{k_e r b}\sum_{n=0}^{\infty}(2n+1)\,d_n\zeta_n(k_e r)\,P_n(\mu).$$

The boundary conditions are now

$$\beta(\Pi_0+\Pi_d) = \beta_i\Pi_i, \qquad (r=a),$$

$$\frac{\partial}{\partial r}(r\Pi_0+r\Pi_d) = \frac{\partial}{\partial r}(r\Pi_i), \qquad (r=a),$$

$$\beta(\Pi_0+\Pi_d) = \beta_e\Pi_e, \qquad (r=c),$$

$$\frac{\partial}{\partial r}(r\Pi_0+r\Pi_d) = \frac{\partial}{\partial r}(r\Pi_e), \qquad (r=c).$$

The equations connecting a_n, b_n, c_n, d_n are

$$(\beta/k)\left[a_n\zeta_n(ka)+\{\zeta_n(kb)+c_n\}\,\psi_n(ka)\right] = (\beta/k_i)\,b_n\psi_n(k_i a),$$

$$a_n\zeta_n'(ka)+\{\zeta_n(kb)+c_n\}\,\psi_n'(ka) = b_n\psi_n'(k_i a),$$

$$(\beta/k)\left[\{a_n+\psi_n(kb)\}\,\zeta_n(kc)+c_n\psi_n(kc)\right] = (\beta_e/k_e)\,d_n\zeta_n(k_e c),$$

$$\{a_n+\psi_n(kb)\}\,\zeta_n'(kc)+c_n\psi_n'(kc) = d_n\zeta_n'(k_e c).$$

On solving these equations we find that the value of the Hertzian function at the surface of the Earth is

$$-\frac{1}{kab}\sum_{n=0}^{\infty}(2n+1)\,g_n P_n(\mu)/h_n,$$

where
$$g_n = \psi_n(kb)\,\zeta_n'(kc)-\psi_n'(kc)\,\zeta_n(kb) - \frac{\beta k_e}{\beta_e k}\frac{\zeta_n'(k_e c)}{\zeta_n(k_e c)}\{\psi_n(kb)\,\zeta_n(kc)-\psi_n(kc)\,\zeta_n(kb)\},$$

$$\begin{aligned} h_n = {} & \psi_n'(ka)\,\zeta_n'(kc)-\psi_n'(kc)\,\zeta_n'(ka) \\ & +\frac{\beta k_i}{\beta_i k}\frac{\psi_n'(k_i a)}{\psi_n(k_i a)}\{\psi_n'(kc)\,\zeta_n(ka)-\psi_n(ka)\,\zeta_n'(kc)\} \\ & -\frac{\beta k_e}{\beta_e k}\frac{\zeta_n'(k_e c)}{\zeta_n(k_e c)}\Big[\psi_n'(ka)\,\zeta_n(kc)-\psi_n(kc)\,\zeta_n'(ka) \\ & +\frac{\beta k_i}{\beta_i k}\frac{\psi_n'(k_i a)}{\psi_n(k_i a)}\{\psi_n(kc)\,\zeta_n(ka)-\psi_n(ka)\,\zeta_n(kc)\}\Big]. \end{aligned}$$

The value of the Hertzian function is consequently dependent on the zeros of $h_{s-\frac{1}{2}}$. If we denote the zero of $h_{s-\frac{1}{2}}$ which corresponds to the zero ν of $F(s)$ by $\nu+\vartheta$ (so that ϑ has not the significance as in §§ 7–8), the equation which determines ϑ approximately is

$$\begin{aligned}\sinh u \sinh v \sinh \{\chi(\nu)+(v-u)\vartheta\} &-(ik/k_i)\sinh v \cosh \{\chi(\nu)+(v-u)\vartheta\} \\ -(ik/k_e)\left(1-\frac{s^2}{k_e{}^2 C^2}\right)^{\frac{1}{2}} &[\sinh u \cosh \{\chi(\nu)+(v-u)\vartheta\} \\ -(ik/k_i) \qquad &[\sinh \{\chi(\nu)+(v-u)\vartheta\}] = 0,\end{aligned}$$

where $\sinh \chi(\nu) = 0$ and $s = \nu+\vartheta$. On solving we find that

$$\begin{aligned}(v-u)\vartheta = &\tanh^{-1}[(ik/k_i)\operatorname{cosech} u] \\ &+\tanh^{-1}\left[(ik/k_e)\left(1-\frac{\nu^2}{k_e{}^2C^2}\right)^{\frac{1}{2}}\operatorname{cosech} v\right].\end{aligned}$$

Hence ϑ consists of two terms each of which is of much the same type as the single term investigated in § 8.

In order to obtain an expression of the form taken empirically by Austin, it is necessary that k_e should be large compared with k, and then the phase of k_e is nearly $-\frac{1}{4}\pi$; and in the case of waves over the sea we obtain Austin's factor 9·6 if

$$0{\cdot}44\,(a/h)\,.\,|(k/k_e)| = 9{\cdot}5\,\lambda^{-\frac{1}{2}},$$

i.e., if

$$0{\cdot}44\,(a/h)\,(2\pi C/\sigma_e)^{\frac{1}{2}} = 9{\cdot}5.$$

This equation gives

$$h^2\sigma_e = 1{\cdot}67\times 10^{11}.$$

Hence Austin's factor is obtained if the reflector has a height of say 100 kilom. and conductivity $1{\cdot}67\times 10^7$. A conductor with this conductivity* is a fairly good conductor so that Austin's result seems to be quite compatible with what is known concerning the upper atmosphere. The factor corresponding to Austin's number 9·6 in the case of waves over dry land is $9{\cdot}6+(1400/h)$. I am not aware of any quantitative experiments having been carried out for such waves, but the increase in A is quite consistent with the well-known fact that electric waves are transmitted much more easily over water than over dry land.

The fact that $h_{s-\frac{1}{2}}$ has a number of zeros with the same imaginary part, so that the Hertzian function contains a number of terms of the same order of magnitude but of different phases, suggest that, in certain circumstances, interference effects may be produced; such effects have actually been observed.

* The measure of this conductivity in ordinary electro-magnetic units is $1{\cdot}44\times 10^{-15}$; the corresponding specific resistance is $6{\cdot}95\times 10^{14}$. For copper the corresponding resistance is about 1600; for fresh water about 2×10^{14}.

10. It is to be concluded that the ionisation theory which results in the upper regions of the atmosphere acting as a reflector of electric waves of long wave-length is sufficient to explain the observed facts concerning the rate of decay of electric waves transmitted over large sheets of water; and it has been seen in my previous paper that a theory of pure diffraction is insufficient to account for these facts. The results contained in this paper consequently afford a confirmation of the theory put forward by Heaviside and others and modified by Eccles.